JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 41, 738–747 (1973)

Absolute Logarithmic Summability of the Fourier Series

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Submitted by R. P. Boas

1. INTRODUCTION

We write $\zeta[f]$ to denote the Fourier series of $f(x)$ where $f(x)$ is periodic with period 2π and belongs to the class L^p, $p > 1$.

The question whether the summability $|C, \delta|$, $0 < \delta \leqslant 1$, at a point, of the Fourier series of $f(x) \in L$, is a local property of $f(x)$ was answered in the negative by Bosanquet and Kestelman in 1939. In 1940 Foà [2] demonstrated with the help of an example that the summability $|C, 1/p|$ of $\zeta[f]$, at a point, is not a local property. The result showing that the summability $|C, \delta|$, $\delta > 1/p$, of $\zeta[f]$, for $1 < p \leqslant 2$, is a local property was first obtained by Tsuchikura [7]. Yano in 1953 framed a very simple example demonstrating the nonlocal nature of the summability $|C, 1/p|$, $1 < p \leqslant 2$ of $\zeta[f]$. In 1954 Tsuchikura [8] constructed another example to the same effect. He also proved that for $p > 2$, summability $|C, \frac{1}{2}|$ of $\zeta[f]$ does not depend on local conditions. Moreover he found that it is not a local property even for the class of continuous functions.

The problem that naturally springs up is that of finding out if the summability $|R, \log n, \delta|$ of $\zeta[f]$, where $\delta = 1/p$ if $f(x) \in L^p$, $1 < p \leqslant 2$, and $\delta = \frac{1}{2}$ if $f(x) \in L^p$, $p \geqslant 2$, can be achieved under local conditions. When $f(x) \in L$ the summability $|R, \log n, 1|$ of the Fourier series of $f(x)$ was found to be nonlocal in nature by Mohanty [5] and Izumi [3], independently. The object of the present paper is to frame examples to demonstrate that the same is true for the summability $|R, \log n, 1/p|$ of $\zeta[f]$.

2.

THEOREM. *If $f(x) \in L^p$, $p > 1$, then the summability $|R, \log n, 1/p|$ of $\zeta[f]$ for $1 < p \leqslant 2$ and the summability $|R, \log n, \frac{1}{2}|$ of $\zeta[f]$ for $p \geqslant 2$, at a point, is not a local property of the function $f(x)$.*

The following lemmas will be required for the proof of our theorem:

LEMMA 1 [4]. *A necessary condition for the summability* $| C, k |$, $k > 0$, *of* $\sum c_n$ *is that* $\sum c_n/n^k$ *converges.*

LEMMA 2 [6]. *If* $\sum c_n$ *is summable* $| R, \lambda(n), k |$ *for* $k \geqslant 0$, *then* $\sum c_n/\{\lambda(n)\}^k$ *is summable* $| R, \mu(n), k |$, *where* $\mu(n) = e^{\lambda(n)}$.

LEMMA 3 [10, Vol. II, p. 129, (6.6)]. *If* $a_1 \geqslant a_2 \geqslant \cdots$, *and* $a_n \to 0$ *as* $n \to \infty$, *then a necessary and sufficient condition that the function* $g(x) = \sum a_n \cos nx$ *should belong to* L^p, $p > 1$, *is that the sum* $\sum a_n{}^p n^{p-2}$ *should converge.*

3. PROOF OF THEOREM

By Lemmas 1 and 2, it is sufficient to frame an even function $f(x) \in L^p$, $p > 1$, which is equal to zero in a finite interval containing the point $x = 0$ and such that

$$\sum \{| a_{4n} | n^{-1/p}(\log n)^{-1/p}\} = \infty \qquad \text{for } f(x) \in L^p, \quad 1 < p \leqslant 2,$$

and

$$\sum \{| a_{4n} | n^{-1/2}(\log n)^{-1/2}\} = \infty \qquad \text{for } f(x) \in L^p, \quad p \geqslant 2,$$

where a_n denotes the nth Fourier cosine coefficient in the expansion of the even function $f(x)$. We frame three examples of such a function for the cases (i) $1 < p < 2$, (ii) $p > 2$, and (iii) $p = 2$, respectively, which amount to the proof of our theorem.

Case (i). $1 < p < 2$.

EXAMPLE. Let

$$f(x) = \sum_{i=c}^{\infty} c_i(x)\, i^{-1/q}(\log i)^{-1/p} (\log \log i)^{-1} \cos(2R_i x), \qquad 1/p + 1/q = 1,$$

where c is a fixed integer > 3, $r_i = i(\log i)^{p-2}$ and R_i denotes $[r_i]$, the integral part of r_i, if $[r_i]$ is even and denotes $[r_i] + 1$ if $[r_i]$ is odd. Also $c_i(x)$ is chosen to be such that $c_i(x) = 1$ for $\pi/2 \leqslant x \leqslant \pi/2 + \pi/(2R_i)$, and $c_i(x) = 0$ everywhere else in $(0, \pi)$.

We now show that $f(x)$, defined as above, $\in L^p$. We have

$$\int_{\pi/2}^{\pi} | f(x)|^p \, dx = \left(\sum_{n=1}^{c-1} + \sum_{n=c}^{\infty}\right) \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} | f(x)|^p \, dx.$$

Now if we suppose that k is an integer such that

$$\pi/(2R_{k+1}) < \pi/\{2(n+1)\} \leqslant \pi/(2R_k)$$

then

$$\sum_{n=c}^{\infty} \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p \, dx$$
$$< \sum_{n=c}^{\infty} \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} \Big\{ \sum_{\nu=c}^{n} \nu^{-1/q}(\log \nu)^{-1/p} (\log\log \nu)^{-1} + \Big| \sum_{\nu=n+1}^{k+1} (\nu^{-1/q}(\log \nu)^{-1/p} (\log\log \nu)^{-1} \cos(2R_\nu x)) \Big| \Big\}^p dx.$$

We have by Minkowski's inequality

$$\Big(\int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p \, dx \Big)^{1/p}$$
$$\leqslant \Big\{ \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} \Big(\sum_{\nu=c}^{n} \nu^{-1/q}(\log \nu)^{-1/p} (\log\log \nu)^{-1} \Big)^p dx \Big\}^{1/p}$$
$$+ \Big\{ \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} \Big| \sum_{\nu=n+1}^{k+1} \nu^{-1/q}(\log \nu)^{-1/p} (\log\log \nu)^{-1} \cos(2R_\nu x) \Big|^p dx \Big\}^{1/p}$$
$$= O\Big[\Big\{ \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} n(\log n)^{-1} (\log\log n)^{-p} \, dx \Big\}^{1/p} \Big]$$
$$+ O\Big[\Big\{ \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} n^{-p/q}(\log n)^{-1} (\log\log n)^{-p} (\log n)^{2p-p^2} x^{-p} \, dx \Big\}^{1/p} \Big]$$
$$= O\{n^{-1/p}(\log n)^{-1/p} (\log\log n)^{-1}\}$$
$$+ O\{n^{-1/q-2/p}(\log n)^{-1/p+2-p} (\log\log n)^{-1}\}.$$

Or

$$\int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p \, dx = O\{n^{-1}(\log n)^{-1} (\log\log n)^{-p}\},$$

so that

$$\sum_{n=c}^{\infty} \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p \, dx = O(1).$$

Also it can be trivially proved that

$$\sum_{n=1}^{c} \Big\{ \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p \, dx = O(1),$$

yielding finally that $f(x) \in L^p$, $1 < p < 2$.

We next prove that $\sum_{n=c}^{\infty} | a_{4n} |/\{n^{1/p}(\log n)^{1/p}\}$ does not converge. Let

$$\alpha_i = i^{-1/q}(\log i)^{-1/p} (\log \log i)^{-1},$$

then we have

$$\begin{aligned} a_{4n} &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos 4nx \, dx \\ &= \frac{1}{\pi} \sum_{i=c}^{\infty} \alpha_i \int_{\pi/2}^{\pi/2+\pi/(2R_i)} 2 \cos(2R_i x) \cos 4nx \, dx \\ &= \frac{1}{\pi} \sum{}' \alpha_i [(2R_i + 4n)^{-1} \sin\{(R_i + 2n + 1) \pi + 2n\pi/R_i\} \\ &\quad + (2R_i - 4n)^{-1} \sin\{R_i - 2n + 1) \pi - 2n\pi/R_i\}] \\ &\quad + \frac{1}{\pi} \sum_{i=\mu}^{\nu} \alpha_i \pi/(2R_i), \end{aligned}$$

where $\sum{}'$ denotes summation for the range $c \leqslant i \leqslant \infty$, excluding the term or terms where $R_i = 2n$, μ and ν being the smallest and largest values of i for which $R_i = 2n$. Obviously

$$\sum_{i=\mu}^{\nu} \alpha_i/R_i > 0.$$

Hence

$$\begin{aligned} a_{4n} &> \frac{1}{\pi} \sum{}' \alpha_i \sin(2n\pi/R_i) \cdot 2n/(R_i^2 - 4n^2) \\ &= -\, 2n/\pi \sum_{i=c}^{\mu-1} \alpha_i (4n^2 - R_i^2)^{-1} \sin(2n\pi/R_i) \\ &\quad + 2n/\pi \sum_{i=\nu+1}^{\infty} \alpha_i (R_i^2 - 4n^2)^{-1} \sin(2n\pi/R_i) \\ &= -\, A + B, \qquad \text{say.} \end{aligned} \tag{3.1}$$

Now for $i \geqslant \nu + 1$, $R_i > 2n$, therefore B always remains positive. Also

$$\begin{aligned} -\, (\pi/2)\, A &= \sum_{R_i \leqslant 2n/\log\log n} O(\alpha_i/n) - \sum_{2n-2 \geqslant R_i > 2n/\log\log n} \alpha_i n (4n^2 - R_i^2)^{-1} \\ &\quad \times \{\sin(2n\pi/R_i) - \sin(2n\pi/(r_i + 2))\} \\ &\quad - \sum_{2n-2 \geqslant R_i > 2n/\log\log n} \alpha_i n (4n^2 - R_i^2)^{-1} \sin(2n\pi/(r_i + 2)) \\ &= O\{n^{-1/q}(\log n)^{-1/q} (\log\log n)^{-1-1/p}\} - A_1 - A_2\,, \qquad \text{say.} \end{aligned} \tag{3.2}$$

Now, since

$$\sin(2n\pi/R_i) - \sin(2n\pi/(r_i + 2)) = O\{n(1/R_i - 1/(r_i + 2))\}$$

for large values of n, we have writing $\delta = r_i + 2 - R_i < 3$,

$$\begin{aligned}
- A_1 &= \sum_{4n/3+2 \geqslant R_i > 2n/\log\log n} O(\alpha_i/R_i^2) - \sum_{2n-2 \geqslant R_i > (4n/3)+2} 2\alpha_i n \\
&\quad \times (4n^2 - R_i^2)^{-1} \sin\{n\pi\delta/(R_i(r_i + 2))\} \cdot \cos\{n\pi\delta(1/R_i + 1/(r_i + 2))\} \\
&= O\{(\log n)^k n^{-1-1/q}\} \\
&\quad - \sum_{2n-2 \geqslant R_i > (4n/3)+2} \{2\alpha_i n(4n^2 - R_i^2)^{-1} \sin\theta_1 \cos\theta_2\},
\end{aligned}$$

where k is a proper constant,

$$\theta_1 = n\pi\delta/\{R_i(r_i + 2)\} \qquad \text{and} \qquad \theta_2 = n\pi(1/R_i + 1/(r_i + 2)).$$

Now for n large, $\sin\theta_1$ is positive. Also for $2n - 2 \geqslant R_i > (4n/3) + 2$,

$$\begin{aligned}
\theta_2 &\leqslant 2n\pi/R_i \\
&< 3\pi/2,
\end{aligned}$$

and for n large

$$\begin{aligned}
\theta_2 &\geqslant 2n\pi/(R_i + 2) \\
&\geqslant \pi.
\end{aligned}$$

Thus $\pi \leqslant \theta_2 < 3\pi/2$, so that $\cos\theta_2$ is negative. Hence $-\sin\theta_1 \cdot \cos\theta_2$ is positive, implying that

$$- A_1 = O\{(\log n)^k n^{-1-1/q}\} + \text{a positive quantity}. \tag{3.3}$$

Next we have

$$\begin{aligned}
- A_2 &= \sum_{2n-2 \geqslant R_i > 2n/\log\log n} \alpha_i n(4n^2 - R_i^2)\{(r_i + 2)^2 (\log i)^{3-p}\} \\
&\quad \times \{2n\pi(\log i - 2 + p)\}^{-1} \{\sin(2n\pi/(r_i + 2))\} (-2n\pi)(r_i + 2)^{-2} \\
&\quad \times \{(\log i - 2 + p)(\log i)^{p-3}\} \\
&= K\{\alpha_i(\log i)^{2-p}(r_i + 2)^2 n^{-1}\}_{R_i=2n-2} \int_{R_i=\xi}^{R_i=2n-2} \sin\theta \, d\theta,
\end{aligned}$$

where $\theta = 2n\pi/(r_i + 2)$ and $2n - 2 > \xi > 2n/\log\log n$, and K is a positive constant.[1] Thus

$$- A_2 = KA_n[-\cos\{2n\pi/(r_i + 2)\}]_{R_i=\xi}^{R_i=2n-2},$$

[1] The constant K will not necessarily be the same at its each occurence.

where

$$A_n = [n^{-1}\alpha_i(\log i)^{2-p}(r_i + 2)^2]_{R_i=2n-2}$$
$$= O[n^{1/p}(\log n)^{-1/q}(\log\log n)^{-1}],$$

and A_n is positive. This gives

$$\begin{aligned} - A_2 &= KA_n[-\cos\{2n\pi/(R_i + \eta + 1)\}]_{R_i=\xi}^{R_i=2n-2}, \qquad \text{where} \qquad 0 \leqslant \eta < 2, \\ &= KA_n[-\cos\{(1-\eta)\,\pi/(2n - 1 + \eta)\} + \cos\{2n\pi/(\eta + \xi + 1)\}] \\ &= KA_n[1 + \cos\{2n\pi/(\eta + \xi + 1)\} + O(1/n^2)] \\ &= O[n^{-1-1/q}(\log n)^{-1/q}(\log\log n)^{-1}] + K_1, \qquad \text{where} \qquad K_1 \geqslant 0. \end{aligned} \tag{3.4}$$

Thus by Eqs. (3.1–3.4), we have

$$\begin{aligned} \sum |a_{4n}|\, n^{-1/p}(\log n)^{-1/p} &> O\left[\sum n^{-1}(\log n)^{-1}(\log\log n)^{-1-1/p}\right] \\ &\quad + O\left[\sum n^{-2}(\log n)^k\right] + \sum K_1 n^{-1/p}(\log n)^{-1/p} \\ &\quad + \sum Bn^{-1/p}(\log n)^{-1/p} \\ &> \sum Bn^{-1/p}(\log n)^{-1/p} + O(1). \end{aligned}$$

Also we have

$$\begin{aligned} &\sum_{n=c}^{\infty} Bn^{-1/p}(\log n)^{-1/p} \\ &\quad = \frac{2}{\pi}\sum_{n=c}^{\infty}\left[n^{-1/p}(\log n)^{-1/p}\left\{\sum_{i=\nu+1}^{\infty}\alpha_i n(R_i^2 - 4n^2)^{-1}\sin(2n\pi/R_i)\right\}\right] \\ &\quad \geqslant \frac{2}{\pi}\sum_{i=4}^{\infty}\left[\alpha_i\left\{\sum_{2n\leqslant R_i-2} n^{1/q}(\log n)^{-1/p}(R_i^2 - 4n^2)^{-1}\sin(2n\pi/R_i)\right\}\right] \\ &\quad > K\sum_{i=4}^{\infty}\alpha_i \sum_{R_i/4<2n\leqslant R_i/2}\{n^{1/q}(\log n)^{-1/p}(R_i^2 - 4n^2)^{-1}\sin(2n\pi/R_i)\} \\ &\quad > K\sum_{i=4}^{\infty}\alpha_i \sum_{R_i/4<2n\leqslant R_i/2}\{n^{1/q}(\log n)^{-1/p}\,n^{-2}\} \\ &\quad > K\sum_{i=4}^{\infty}\{\alpha_i(\log R_i)^{-1/p}(R_i)^{-1/p}\} \\ &\quad > K\sum_{i=4}^{\infty}[\{i^{-1/q}(\log i)^{-1/p}(\log\log i)^{-1}\}\cdot\{i^{-1/p}(\log i)^{1/p-1}\}] \\ &\quad > K\sum_{i=4}^{\infty} i^{-1}(\log i)^{-1}(\log\log i)^{-1} \\ &\quad = \infty. \end{aligned}$$

This completes the proof of our theorem for the case $1 < p < 2$.

Case (ii). $p > 2$.

EXAMPLE. Let an even function $f(x)$ be defined as

$$f(x) = \sum_{i=c}^{\infty} c_i(x)\, i^{-1/q}(\log i)^{-1/2} (\log \log i)^{-1} \cos(2R_i x), \qquad c > 3,$$

where $R_i = [r_i]$ if $[r_i]$ is even and $R_i = [r_i] + 1$ if $[r_i]$ is odd, $[r_i]$ being the integral part of r_i where $r_i = i^{2/p}$. Also $c_i(x) = 1$ for $\pi/2 \leqslant x \leqslant \pi/2 + \pi/(2R_i)$ and $c_i(x) = 0$ elsewhere in $(0, \pi)$.

We first show that $f(x) \in L^p$, $p > 2$. We have

$$\int_{\pi/2}^{\pi} |f(x)|^p\, dx = \sum_{n=1}^{\infty} \int_{\pi/2+\pi/(2n+1)}^{\pi/2+\pi/(2n)} |f(x)|^p\, dx.$$

Now supposing that k is an integer such that

$$\pi/(2R_{k+1}) < \pi/\{2(n+1)\} \leqslant \pi/(2R_k)$$

we have proceeding as before

$$\left\{\int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p\, dx\right\}^{1/p}$$

$$< \left[\int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} \left\{\sum_{\nu=c}^{n} \nu^{-1/q}(\log \nu)^{-1/2} (\log \log \nu)^{-1}\right.\right.$$

$$\left.\left. \times \left| \sum_{\nu=n+1}^{k+1} \nu^{-1/q}(\log \nu)^{-1/2} (\log \log \nu)^{-1} \cos(2R_\nu x) \right|\right\}^p dx\right]^{1/p}$$

$$= O[\{(\log n)^{p/2} (\log \log n)^p\, n\}^{-1/p}].$$

Thus

$$\sum_{n=c}^{\infty} \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p\, dx = O\left[\sum_{n=c}^{\infty} n^{-1}(\log n)^{-p/2} (\log \log n)^{-p}\right]$$

$$= O(1).$$

Also it is easily verified that

$$\sum_{n=1}^{c-1} \int_{\pi/2+\pi/\{2(n+1)\}}^{\pi/2+\pi/(2n)} |f(x)|^p\, dx = O(1),$$

showing that $f(x) \in L^p$.

We next show that $\sum \{| a_{4n} | n^{-1/2}(\log n)^{-1/2}$ does not converge. Let

$$\alpha_i = i^{-1/q}(\log i)^{-1/2} (\log \log i)^{-1}.$$

We have

$$\begin{aligned}
a_{4n} &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos 4nx \, dx \\
&= \frac{1}{\pi} \sum_{i=c}^{\infty} \alpha_i \int_{\pi/2}^{\pi/2+\pi/(2R_i)} 2 \cos(2R_i x) \cdot \cos(4nx) \, dx \\
&> - 2n/\pi \sum_{i=c}^{\mu-1} \alpha_i \sin(2n\pi/R_i) \cdot (4n^2 - R_i^2)^{-1} \\
&\quad + \frac{2n}{\pi} \sum_{i=\nu+1}^{\infty} (R_i^2 - 4n^2)^{-1} \sin(2n\pi/R_i) \\
&= - A + B, \qquad \text{say.}^2
\end{aligned}$$

Now following the analysis used in the proof of Case (i), we obtain B to be positive, and

$$- A = \text{a positive quantity} + O[n^{-1/2}(\log n)^{-1/2} (\log \log n)^{-1-1/p}].$$

Thus

$$\begin{aligned}
&\sum_{n=c}^{\infty} | a_{4n} | n^{-1/2}(\log n)^{-1/2} \\
&\qquad > \sum_{c}^{\infty} B n^{-1/2}(\log n)^{-1/2} + O\left[\sum_{c}^{\infty} n^{-1}(\log n)^{-1} (\log \log n)^{-1-1/p}\right] \\
&\qquad > \sum_{c}^{\infty} \alpha_i \sum_{R_i/4<2n<R_i/2} n^{1/2}(\log n)^{-1/2} (R_i^2 - 4n^2)^{-1} \sin(2n\pi/R_i) + O(1) \\
&\qquad > K \sum_{c}^{\infty} \alpha_i (R_i \log i)^{-1/2} + O(1) \\
&\qquad > K \sum_{c}^{\infty} i^{-1}(\log i)^{-1} (\log \log i)^{-1} + O(1) \\
&\qquad = \infty.
\end{aligned}$$

This gives the proof of the theorem for the Case (ii).

[2] μ and ν are integers as defined in the proof of case (i).

Case (iii). $p = 2$.

EXAMPLE. Let $f(x)$ be an even function defined as

$$f(x) = \sum_{n=c}^{\infty} n^{-1/2}(\log n)^{-1/2}(\log\log n)^{-1}\cos 2nx \qquad \text{for } \pi/2 < x \leqslant \pi, \quad c > 3,$$

and $f(x) = 0$ everywhere else in $(0, \pi)$.

Applying Lemma 3 we find that $f(x)$, defined as above, belongs to the class L^2. Also

$$a_{2\nu} = \frac{2}{\pi}\int_{\pi/2}^{\pi}\left\{\cos 2\nu x \sum_{n=c}^{\infty} n^{-1/2}(\log n)^{-1/2}(\log\log n)^{-1}\cos 2nx\right\} dx$$

$$= \tfrac{1}{2}(\nu\log\nu)^{-1/2}(\log\log\nu)^{-1},$$

since the termwise integration is permitted by a well known result (cf. Zygmund 10, Vol. I, p. 59). Thus

$$\sum_{\nu=c}^{\infty} | a_\nu | \nu^{-1/2}(\log\nu)^{-1/2} \geqslant \sum_{\nu=c}^{\infty} | a_{2\nu} | (2\nu)^{-1/2}(\log 2\nu)^{-1/2}$$

$$= K\sum_{\nu=c}^{\infty} \nu^{-1}(\log\nu)^{-1}(\log\log\nu)^{-1}$$

$$= \infty.$$

Now by Lemmas 1 and 2, the necessary condition for the summability $| R, \log n, \frac{1}{2} |$ of the Fourier series of $f(x)$ being not satisfied, we obtain the proof of the theorem for the Case (iii).

The proof of our theorem is now complete.

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